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POISSON SUMMATION FORMULAS FOR GROUPS-1: FINITE GROUPS

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PREFACE

As a part of its research program, The RAND Corporation engages in basic supporting studies in mathematics. Progress in one branch of mathematics often results from observing analogies to known results in another branch. The Poisson summation formula has long been a basic tool in mathematical analysis. The present research extends the idea of this formula to make it applicable to certain mathematical structures (groups) that are of basic importance in many fields of mathematics.

SUMMARY

A summation formula for finite groups is established analogous in form and proof to the classical Poisson summation formula.

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POISSON SUMMATION FORMULAS FOR GROUPS—I: FINITE GROUPS

1. INTRODUCTION

The classical Poisson summation formula asserts that under appropriate conditions concerning the function $f(x)$, we have the relation

$$(1.1) \qquad \sum_{-\infty<n<\infty} f(n) = \sum_{-\infty<n<\infty} \left(\int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\, dx \right).$$

A formal derivation which shows the group-theoretic origin of this formula is the following. Consider the function

$$(1.2) \qquad F(y) = \sum_n f(n + y),$$

periodic with period 1. Expanding $F(y)$ in a Fourier series, we have

$$(1.3) \qquad F(y) = \sum_m a_m e^{2\pi imy},$$

where the coefficients are determined in the following way:

$$\begin{aligned}(1.4) \qquad a_m &= \int_0^1 F(y)e^{-2\pi imy}\, dy \\ &= \sum_{-\infty<n<\infty} \int_0^1 f(n + y)e^{-2\pi imy}\, dy \\ &= \int_{-\infty}^{\infty} f(y)e^{-2\pi imy}\, dy.\end{aligned}$$

Setting $y = 0$, we have (1.1). To obtain group-theoretic versions of (1.1), we need only to interpret each of these steps in a suitable fashion.

Since the Poisson summation formula can be used to obtain such fundamental manifestations of duality as the transformation formulas for the theta functions and Gauss sums, it is to be expected that the corresponding formulas for groups could be used equally to obtain duality theorems for groups. These matters and generalizations of the results obtained here will be discussed subsequently.

2. FINITE-GROUP VERSION

Let H be a finite group of order N, and G a proper subgroup of order M. Let $x_1, x_2, \ldots, x_M$ be the elements of G, and let y denote an element of H. Further, let $f(y)$ be defined for $y \in H$.

Let $\{X_i(y)\}$ be a complete set of characters with the property that any function defined on H can be expanded in the form $\sum_i a_i X_i(y)$. Then we have the following result.

Theorem.

$$\frac{1}{M} \sum_i f(x_i) = \frac{1}{N} \sum_{X_G} \left[\sum_y f(y) X(y^{-1}) \right], \tag{2.1}$$

where X_G denotes the set of characters satisfying the relation

$$X(x_i) = 1 \quad \textit{for} \quad x_i \in G. \tag{2.2}$$

3. PROOF

To establish the foregoing result, we consider the character expansion of the function

$$(3.1) \qquad F(y) = \sum_i f(x_i y).$$

Writing

$$(3.2) \qquad F(y) = \sum_j a_j X_j(y),$$

we see that the invariance, $F(y) = F(yx)$ for $x \in G$, requires that

$$(3.3) \qquad a_j(1 - X_j(x)) = 0.$$

Hence $a_j = 0$ if $X_j(x) \neq 1$ for $x \in G$.

The orthogonality property of the characters yields the representation for the coefficients:

$$(3.4) \qquad a_j = \frac{1}{N} \sum_y F(y) X_j(y^{-1}).$$

Using the expression for $F(y)$, we have

$$\begin{aligned}(3.5) \qquad a_j &= \frac{1}{N} \sum_y \left(\sum_i f(x_i y) \right) X_j(y^{-1}) \\ &= \frac{1}{N} \sum_i \left(\sum_y f(x_i y) X_j(y^{-1}) \right) \\ &= \frac{1}{N} \sum_i X_j(x_i^{-1}) \left(\sum_y f(y) X_j(y^{-1}) \right) \\ &= \frac{M}{N} \sum_y f(y) X_j(y^{-1}),\end{aligned}$$

since $X_j(x_i^{-1}) = 1$ for x_i and the admissible j-values.

Hence

$$(3.6) \qquad \sum_i f(x_i y) = \frac{M}{N} \sum_j \left(\sum_{y_1} f(y_1) X_j(y_1^{-1}) \right) X_j(y).$$

Setting $y = I$, the identity element, we obtain the stated result.

Observe that each step has been the analogue of the corresponding step in Sec. 1.